REMARKS ON THE DISJOINT MINIMAL GRAPHS CONJECTURE

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ABSTRACT. It is known that there can be at most 3 disjoint minimal graphs in R^3 with 0 boundary values, and conjectured that there can be at most 2. In this paper we explore properties that the graphs would have if there were 3.

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1. INTRODUCTION

Let D be an unbounded domain in $\mathbb{R}^2$. We are interested in solutions to the minimal surface equation for the boundary value problem

$$Lu = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{1.1}$$
$$u > 0 \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D.$$

Henceforth we shall use complex notation $z = x + iy$ for convenience.

In his talk at the Clay Mathematics Institute's Summer School on "The Global Theory of Minimal Surfaces" at MSRI in the summer of 2001, Meeks conjectured that there can be at most two such solutions over disjoint domains.

In his paper [9], Spruck proved this under several stringent side conditions on the behavior of u. Later, without any side conditions, Tkachev [11] proved that there could be at most three such disjoint graphs.

We will consider the possible case that there are 3 graphs when the domains are simply connected, and provide information regarding the necessary properties solutions would have in this case. While these properties are substantial, they are not sufficient to resolve the conjecture.

With u as in (1.1), let $M(r,u) = \max\limits_{z\in\{|z|=r\}\cap D} u(z)$. Then we have the following.

Theorem 1.. *Let u_1, u_2, u_3 be solutions to (1.1) over disjoint simply connected domains D_1, D_2, D_3 respectively. Then for sufficiently large r,there exist positive constants κ_1 and κ_2 such that*

$$\kappa_1 r < M(r,u_j) < \kappa_2 r \qquad (j = 1,2,3). \tag{1.2}$$

We note that for sufficiently large c,, by replacing the u_j by $u_j - c$ in the subdomains $u_j > c$ and using the theorem of Tkachev, we may assume that each D_j is bounded by a real analytic Jordan arc.

2. PRELIMINARIES

Let u be a solution to (1.1) over a domain D bounded by an unbounded differentiable Jordan arc. We shall make use of the parametrization of a surface given by u in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), U(\zeta))$ with ζ in the parameter space. Our notation will then be given by

$$f(\zeta) = x(\zeta) + iy(\zeta). \tag{2.1}$$

Note that we shall use $\zeta = \sigma + i\tau$ in the parameter space, and $z = x + iy$ in D.

Then $f(\zeta)$ is univalent and harmonic, and since D is simply connected it can be written in the form

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)} \tag{2.2}$$

where $h(\zeta)$ and $g(\zeta)$ are analytic, and

$$|h'(\zeta)| > |g'(\zeta)|. \tag{2.3}$$

We dismiss the trivial case $g' \equiv 0$ so that

$$U(\zeta) = \pm\, 2\Re e\, i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta. \tag{2.4}$$

(cf. [2, §10.2]).

If the parameter space is taken to be the right half plane H, then $z = f(\zeta)$, $u(f(\zeta)) = U(\zeta)$, and $U(\zeta)$ is harmonic and positive in H and vanishes on ∂H. Thus, (cf. [10, p. 151]),

$$U(\zeta) = K\, \Re e\, \zeta, \tag{2.5}$$

where K is a positive constant. This with (2.4) gives

$$g'(\zeta) = -\frac{C}{h'(\zeta)}$$

where C is a positive constant. The letters $C,\ K,\ B,\ \kappa$ will all be used throughout to denote constants that may depend on u.

By reparametrizing we can make

$$U(\zeta) = 2\Re e\, \zeta \ \text{ and } \ g'(\zeta) = -1/h'(\zeta) \tag{2.6}$$

and then the *analytic dilatation* $a(\zeta)$ is

$$a(\zeta) = \overline{f_{\bar\zeta}(\zeta)}/f_\zeta(\zeta) = -1/h'(\zeta)^2. \tag{2.7}$$

Furthermore, from (2.3) we have, in particular, that

$$|h'(\zeta)| = 1/|g'(\zeta)| > 1. \tag{2.8}$$

For computations in H we shall denote the subsectors by

$$S_\alpha = \{\zeta : |\arg \zeta| < \alpha/2\}.$$

With these preliminaries we have the following.

Theorem 2. *Let u represent one of the solutions in Theorem 1 and parametrized in H as above. For any $\alpha < \pi/2$, let D_α denote the image of S_α. Then in D_α, the laplacian of u tends to zero as*

$$|\Delta u(z) < K/|z| \tag{2.9}$$

for some $K = K(u, \alpha)$ and $|z|$ sufficiently large .

In addition, given $\varepsilon > 0$, with $|\alpha|$ sufficiently close to $\pi/2$ we also have

$$u(z) < \varepsilon|z| \qquad z \in \partial D_\alpha \tag{2.10}$$

for sufficiently large $|z|$.

The proof of Theorem 2, which will be carried out in §3-§10 uses the fact that, with (2.8), $\log|h'(\zeta)|$ is a positive harmonic function and therefore has a Poisson representation.

Lemma A. *With $h(\zeta)$ as in (2.2) and (2.8),*

$$\log|h'(\zeta)| = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} \frac{\log|h'(is)|ds}{|\zeta - is|^2} \qquad \zeta \in H. \tag{2.11}$$

Proof. Since by (2.8) $\log|h'|$ is a positive harmonic function in H it can be represented (cf. [10, p. 149])

$$\log|h'(\zeta)| = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} \frac{\log|h'(is)|ds}{|\zeta - is|^2} + c\,\zeta \qquad c \geq 0.$$

Furthermore, by [10, Theorem IV. 19.], $\log h'(\zeta)/\zeta \to c$ and $h''(\zeta)/h'(\zeta) \to c$ in S_ϵ. By [12, Lemma 2] it follows that $c = 0$.

□

We shall also need the *conjugate function*, given in D by (e.g. ([6, p. 62])

$$v(z) = \int -\frac{u_{w_2}}{\sqrt{1+|\nabla u|}}dw_1 + \frac{u_{w_1}}{\sqrt{1+|\nabla u|}}dw_2. \tag{2.12}$$

As in (2.6), in the parameter half plane H we may take this as

$$V(\zeta) = 2\Im m\,\zeta. \tag{2.13}$$

3. Miklykov's method

We shall take advantage of the work of V. Mikljukov [6] on the moduli of path families on minimal surfaces.

Let D be a simply connected unbounded domain in $\mathbb{R}^2$. Let F denote the surface given by $u(z), \quad z = x_1 + ix_2 \in D$ with

$$ds_F^2 = (1 + u_{x_1}^2)dx_1^2 + 2u_{x_1}u_{x_2}dx_1dx_2 + (1 + u_{x_2}^2)dx_2^2$$

and

$$d\sigma_F = \sqrt{1 + |\nabla u|^2}\, dx_1dx_2$$

the respective length and area elements for F.

For a family Γ of curves in D we define the *modulus* of Γ in the metric of F by

$$\mathrm{mod}_F\Gamma = \inf \iint_D \rho^2(z)d\sigma_F,$$

the inf being taken over all nonnegative measurable functions ρ on D satisfying

$$\inf_{\gamma\in\Gamma} \int_\gamma \rho(z)ds_F \geq 1.$$

The utility of the modulus comes from the elementary observation that it is a conformal invariant (cf. [6, p.65]).

We shall use estimates on the modulus for path families of curves on a surface F given by solutions $u(z)$ to the minimal surface equation over domains D as in (1.1). Corresponding to this we also have the conformal parametric description given by (2.1) -(2.8).

Path families Γ in H correspond to path families on F which project to path families $f(\Gamma)$ in D. By conformal invariance, the modulus may be computed either in H, or with the surface metric in D.

Let $D(r, R) = D \cap \{r < |z| < R\}$. Fix a and b on ∂D. For sufficiently large $t > 0$ let $S(t)$ be the component of $D \cap \{z : |z| = t\}$ separating a from ∞ in D. Choosing r large enough so $S(r)$ separates b from ∞ in D, and $R > r$, let T be the subdomain of D between $S(r)$ and $S(R)$. Let $\Gamma = \Gamma(r, R)$ be a family of curves in T that join $S(r)$ and $S(R)$.

In what follows, we shall extend the results of [6] slightly to suit our needs. We assume there are sequences $\{r_m\}, \{r'_m\}, \{r''_m\}$ tending to ∞ such that

$$r_m/r'_m \to \infty \text{ and } r''_m/r_m \to \infty. \tag{3.1}$$

Lemma 1. *With the above notations, if for some $C > 0$,*

$$u(r_mz)/(r_mz) \leq C \quad r'_m \leq |r_mz| \leq r''_m \quad r_mz \in D, \tag{3.2}$$

and $M_n = \max_{S(r_n'))}(|z|/r_n|, r_n u(z/r_n))$ *then*

$$\mathrm{mod}_F\Gamma(r_n', r_n'') \le \frac{\pi}{2(\log r_n'')^2}\left(\iint\limits_{D(r_n', r_n'')} \frac{dx_1 dx_2}{|z|^2}) + 2\pi^2 + o(1)\right) \ (m \to \infty). \tag{3.3}$$

With the stronger condition

$$u(r_m z)/(r_m z) \to 0 \quad r_m' \le |r_m z| \le r_m'' \quad r_m z \in D, \tag{3.4}$$

then

$$\mathrm{mod}_F\Gamma(r_n', r_n'') \le \frac{1}{(\log r_n'')^2}\left(\iint\limits_{D(r_n', r_n'')} \frac{dx_1 dx_2}{|z|^2}) + 2\pi^2 + o(1)\right) \ (m \to \infty). \tag{3.5}$$

Proof. For notational simplicity we shall replace $z/r_m,\ r_m'/r_m,\ r_m''/r_m,\ u(r_m z)/r_m,$ and M_n by $z,\ r,\ R,\ u(z)$, and M respectively.

Following Mikljukov, we choose the density function

$$\rho(z) = (|z|^2 + u^2(z))^{-1/2},$$

for $z \in B = \overline{T} \cap \overline{D(r, R)}$ and $\rho(z) = 0$ for all the remaining values $z \in D$. Hence

$$\mathrm{mod}_F\Gamma \le \frac{\iint_B (|z|^2 + u^2(z))^{-1} d\sigma_F}{\left(\inf\limits_{\gamma\in\Gamma} \int_\gamma (|z|^2 + u^2(z))^{-1/2} ds_F\right)^2}. \tag{3.6}$$

For the denominator in (3.6), let γ be a curve in Γ. Then

$$\int_\gamma \rho\, ds_F = \int_\gamma (|z|^2 + u^2(z))^{-1/2} ds_F \ge \log\left(1 + \frac{R-r}{M}\right). \tag{3.7}$$

To obtain a bound for the numerator we let

$$\eta(z) = \frac{1}{|z|}\ \arctan\delta(z), \qquad \delta(z) = \frac{u(z)}{|z|}, \tag{3.8}$$

so that

$$\frac{1}{|z|^2 + u^2(z)} = \frac{1}{|z|^2(1 + \delta^2(z))}. \tag{3.9}$$

To begin with, we note that

$$\eta_{x_i} = \frac{-x_i}{|z|^3}\ \arctan\delta(z) + \frac{1}{|z|}\frac{1}{1+\delta^2(z)}\left(\frac{u_{x_i}}{|z|} - \frac{x_i u}{|z|^3}\right),$$

and so

$$\sum_{i=1}^{2} \frac{u_{x_i}^2}{|z|^2(1+\delta^2(z))\sqrt{1+|\nabla u|^2}} = \sum_{i=1}^{2} \frac{\eta_{x_i} u_{x_i}}{\sqrt{1+|\nabla u|^2}} \tag{3.10}$$

$$+\frac{\arctan \delta(z)}{|z|^3} \sum_{i=1}^{2} \frac{x_i u_{x_i}}{\sqrt{1+|\nabla u|^2}} + \frac{1}{|z|^3} \frac{\delta(z)}{1+\delta^2(z)} \sum_{i=1}^{2} \frac{x_i u_{x_i}}{\sqrt{1+|\nabla u|^2}}.$$

Also,

$$\sqrt{1+|\nabla u|^2} = \frac{1}{\sqrt{1+|\nabla u|^2}} + \sum_{i=1}^{2} \frac{u_{x_i}^2}{\sqrt{1+|\nabla u|^2}},$$

and thus

$$\sum_{i=1}^{2} \frac{u_{x_i}^2}{|z|^2(1+\delta^2(z))\sqrt{1+|\nabla u|^2}} = \frac{\sqrt{1+|\nabla u|^2}}{|z|^2(1+\delta^2(z))} \tag{3.11}$$

$$-\frac{1}{|z|^2(1+\delta^2(z))\sqrt{1+|\nabla u|^2}}.$$

Finally, by Green's Theorem

$$\iint_B \sum_{i=1}^{2} \frac{\eta_{x_i} u_{x_i}}{\sqrt{1+|\nabla u|^2}}\, dx_1 dx_2 + \iint_B \eta \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\, dx_1 dx_2 \tag{3.12}$$

$$= \int_{\partial B} \frac{\eta \nabla u \cdot n}{\sqrt{1+|\nabla u|^2}} |dz|,$$

with n being the outer unit normal. The middle term vanishes because of the minimal surface equation.

Using (4.5)–(4.7) in (4.8) we then have

$$\iint_B \frac{d\sigma_F}{|z|^2+u^2(z)} = \iint_B \frac{1}{(1+\delta^2(z))\sqrt{1+|\nabla u|^2}} \frac{dx_1 dx_2}{|z|^2}$$

$$+ \int_{\partial B} \frac{\eta \nabla u \cdot n}{\sqrt{1+|\nabla u|^2}} |dz| + \iint_B \arctan \delta(z) \sum_{i=1}^{2} \frac{x_i u_{x_i}}{|z|\sqrt{1+|\nabla u|^2}} \frac{dx_1 dx_2}{|z|^2}$$

$$+ \iint_B \frac{\delta(z)}{1+\delta^2(z)} \sum_{i=1}^{2} \frac{x_i u_{x_i}}{|z|\sqrt{1+|\nabla u|^2}} \frac{dx_1 dx_2}{|z|^2}$$

Since

$$\sum_{i=1}^{2} \frac{x_i u_{x_i}}{|z|\sqrt{1+|\nabla u|^2}} \le \frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}},$$

this gives

$$\iint_B \frac{d\sigma_F}{|z|^2+u^2(z)} \le \iint_B \frac{1}{(1+\delta^2(z))\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}$$
$$+\int_{\partial B}\frac{\eta\nabla u\cdot n}{\sqrt{1+|\nabla u|^2}}\frac{|dz|}{|z|}+\iint_B \arctan\delta(z)\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}$$
$$+\iint_B \frac{\delta(z)}{1+\delta^2(z)}\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}}\frac{dx_1dx_2}{|z|^2}.$$

Since

$$\int_{\partial B}\frac{\eta\nabla u\cdot n}{\sqrt{1+|\nabla u|^2}}\frac{|dz|}{|z|}\le\int_{\partial B}\frac{\eta|\nabla u|}{\sqrt{1+|\nabla u|^2}}\frac{|dz|}{|z|}\le 2\pi^2,$$

we have

$$\iint_B \frac{d\sigma_F}{|z|^2+u^2(z)} \le \iint_B \frac{1}{\sqrt{1+\nabla u|^2}}(\frac{1}{1+\delta^2(z)}$$
$$+|\nabla u|\arctan\delta(z)+\frac{|\nabla u|\delta(z)}{1+\delta^2(z)})(1+o(1))\frac{dx_1dx_2}{|z|^2}+2\pi^2 \quad (R\to\infty).$$

By letting $\delta(z)\to\infty$ and using (3.2) we obtain (3.3). Using (3.4) and lettting $\delta\to 0$ we obtain (3.5).

□

We now use the conformal invariance of the mapping $H\to F$ as described in §3 together with the estimate of Lemma 1. Continuing the convention of using $z,\ r,\ R,\ u(z)$, and M for $z/r_m,\ r_m'/r_m,\ r_m''/r_m,\ u(r_mz)/r_m$, and M_n respectively, with $a,\ b,\ f(\zeta)$ as in §3, and continuing with [6, p.67], we take $r>0$ so that $S(r)$ separates b and ∞ in D. For $t\ge r$, let $S^*(t)=f^{-1}(S(t))$ so that $\overline{S^*(t)}$ has endpoints on ∂H in the ζ plane. Let $l(t)$ denote the Jordan curve formed by $\overline{S^*(t)}$ along with its reflection across ∂H and G the annular domain between $l(r)$ and $l(R)$. Let $\tilde\Gamma(r,R)$ be the family of curves separating $l(r)$ and $l(R)$ in G. Then, as in ([6, p.67]), since $l(r)$ and $l(R)$ separate 0 $(=f^{-1}(a))$ and 1 $(=f^{-1}(b))$ from ∞, the modulus (in the Euclidean metric) satisfies [5, pp. 32,56 and 61 (2.10)],

$$\operatorname{mod}\tilde\Gamma(r,R)\le\frac{1}{2\pi}\log(16(P+1)),$$

where

$$P=\min_{\zeta\in l(R)}|\zeta|.$$

Now let $\Gamma^*(r,R)$ be the curves joining $l(r)$ and $l(R)$ in G. Then $\operatorname{mod}\Gamma^*(r,R)=1/\operatorname{mod}\tilde\Gamma(r,R)$. Therefore,

$$\operatorname{mod}\Gamma^*(r,R)\ge\frac{2\pi}{\log(16(m(R)+1))}. \tag{3.13}$$

Let

$$m(t) = \min_{\substack{|z|=t \\ z\in D}} |\zeta(z)|. \tag{3.14}$$

Then by (3.13), the symmetry principle [1, p.16], and conformal invariance,

$$\mathrm{mod}_F \Gamma(r,R) \geq \frac{\pi}{\log(16(m(R)+1))}. \tag{3.15}$$

Thus, (3.15) taken together with Lemma 1 yields

Lemma 2. *With the above notation and conventions, then with (3.2) we have*

$$\liminf_{m\to\infty} \frac{\log m(r''_m)}{(\log r''_m)^2} \Big(\iint_{D(r'_m, r''_m)} (1+o(1)) \frac{dx_1 dx_2}{|z|^2} + 2\pi^2 \Big) \geq 2, \tag{3.16}$$

and if (3.4) holds, then

$$\liminf_{m\to\infty} \frac{\log m(r''_m)}{(\log r''_m)^2} \Big(\iint_{D(r'_m, r''_m)} (1+o(1)) \frac{dx_1 dx_2}{|z|^2} + 2\pi^2 \Big) \geq \pi. \tag{3.17}$$

In using Lemma 2, we shall need an estimate on the quantity $m(t)$.

Lemma 3. *Let $u(z)$ be as in Lemma 1 and Lemma 2. then $M(t) \geq m(t)$.*

Proof. If ζ_0 gives the minimum in (3.14), then $m(t) = |\zeta_0|$. If z_1 is a point on $|z| = t$ where $M(t)$ is achieved and $f(\zeta_1) = z_1$, then by (2.6), $u(z_1) = u(f(\zeta_1)) = \mathrm{Re}\, \zeta_1 \geq |\zeta_0|$ or, in other words, $M(t) \geq m(t)$. □

4. GROWTH LEMMAS

For a given $u(\zeta)$ as in the statement of Theorem 1 over a domain D, we define $\Theta(r) = \Theta(u, r)$ to be the angular measure of the set $D \cap \{|\zeta = r\}$.

Lemma 4. *Let D be an unbounded simply connected domain with with $\Theta(r_n) \leq \beta < \pi$ for an increasing sequence $r_n \to \infty$. If u is a solution to the boundary value problem (1.1). then there exists a sequence $r'_n < r_n$ such that $r_n/r'_n \leq B$ for some $B > 0$. and a constant $K = K(\beta, B)$, such that*

$$u(z) \leq K r_n \qquad z \in D \cap \{|z| = r'_n\}.$$

In case r_{n+1}/r_n remains bounded, it then follows readily that

$$M(r, u) < Cr \tag{4.1}$$

for some $C > 0$ and all large r.

Proof. Let γ be an arc on the unit disk having length β and ℓ be the line segment joining the endpoints of β. Let V be the subset of the disk having its boundary as ℓ and the complement of β with respect to the boundary of the disk.

Extend ℓ symmetrically to length 1, and let $\tilde{\ell}$ be the line formed by translating the extended ℓ parallel to itself 10 units in the direction of the origin. Now construct a rectangle R by joining the endpoints of the extended ℓ to the endpoints of $\tilde{\ell}$. Then $V \subset R$ and it follows from [4, p. 322] that there exists a solution $v(z)$ to the minimal surface equation in R having boundary values $+\infty$ on the extended ℓ and $\tilde{\ell}$, and 0 on the other two sides.

Let η be the distance from the origin to ℓ and $\Delta = \{|z| < \eta/2\}$. Let M denote the maximum of $v(z)$ in Δ.

Now, for any given r_n, the function $u(r_n z)/r_n$ satisfies the minimal surface equation and we consider any of the components of the set where $u(r_n z) > 0$ in $|z| < 1$. In any such component $u(r_n z)/r_n$ is bounded by the corresponding $v(z)$ with the arc γ referred to above, being an arc containing the portion of the component on the boundary. Thus, if the component extends to Δ, then for those points in Δ we have $u(r_n z)/r_n \leq M$. Applying this to all such components we then get $u(z) \leq M r_n$ for $z \in D \cap \{|z| = \eta\, r_n/2\}$ from which the result follows.

□

Lemma 5. *Let u satisfy (1.1) and assume that for sequences $\{r'_n\}$ and $\{r''_n\}$ satisfying $r''_n/r'_n \to \infty$, and that (4.1) holds for $r'_n < |z| < r''_n$. Then,*

$$\liminf_{n\to\infty} \frac{1}{\log r''_n}(1 + o(1)) \iint_{D(r'_n, r''_n)} \frac{dr d\theta}{r} \geq 2, \tag{4.2}$$

and if (3.4) holds, then

$$\liminf_{n\to\infty} \frac{1}{\log r''_n}(1 + o(1)) \iint_{D(r'_n, r''_n)} \frac{dr d\theta}{r} \geq \pi, \tag{4.3}$$

Proof. This follows directly from Lemma 3 and Lemma 4. □f

5. Proof of (1.2)

We first prove the right side of (1.2).

Suppose first that there is a sequence $\{r'_n\}$ tend to ∞ such that

$$M(u_1, r'_n)/r'_n = a_n \to \infty \quad \text{and} \quad M(u_2, r'_n)/r'_n = b_n \to \infty,$$

and take $d_n = \min(a_n, b_n)$. Then, if $r''_n = r'_n/d_n^{3/4}$, we have

$$M(u_1, r)/r \to \infty \quad \text{and} \quad M(u_2, r)/r \to \infty \tag{5.1}$$

for $r'_n < r < r''_n$ as $n \to \infty$. It follows from Lemma 4 that for $\varepsilon > 0$, $\Theta(u_1, r) > \pi - \varepsilon$ and $\Theta(u_2, r) > \pi - \varepsilon$ for $r \in I_n = (r_n'^{1/4}, r_n'^{1/2})$ and n large. This means that $\Theta(u_3, r) < \pi/2$ on I_n so that repeated use of Lemma 4 applies, and hence so does

Lemma 5. However, this is inconsistent with the fact that $\Theta(u_1, r)+\Theta(u_2, r) > 2\pi-2\varepsilon$ for this interval.

This limits the possibility down to only one of the solutions, say u_1 having infinite sequence $\{r'_n\}$ and $\{r''_n\}$ as above tending to ∞ such that $u_1(z)/|z| \to \infty$ for $r'_n < |z| < r''_n$, but u_2 and u_3 satisfy (4.1) on these intervals. This possibility then gives $\Theta(u_1, r) > \pi-\varepsilon$ on I_n and u_2 and u_3 satisfying (4.2) which again is impossible. Thus, each u_j satisfies (4.1) in D_j.

Having established the right side of (1.2), we focus on one of the solutions, say u_1, and use the fact that u_2 and u_3 satisfy (4.1) and hence also (4.2).

Now if there exists a sequence $r_n \to \infty$ on which $M(u_1, r_n)/r_n = \delta_n \to 0$, then $M(u_1, r)/r \to 0$ on the interval $[\sqrt{\delta_n} r_n, r_n]$ so using this interval for u_1 in (4.3) and also for u_2 and u_3 in (4.2) we obtain a contradiction thus proving the left side of (1.2). □

6. THE THREE CASES

In what follows we assume 3 disjoint solutions $u_1,\ u_2,\ u_3$ as in Theorem 1. By §5 we may apply Lemma 5 to u_2 and u_3. As a consequence, it must then be that for the sequences $\{r'_n\}$ and $\{r''_n\}$ with $r''_n/r'_n \to \infty$, u_1 in its domain D_1 must satisfy

$$\limsup_{n\to\infty} \frac{1}{\log r''_n}(1 + o(1)) \iint_{D_1(r'_n, r''_n)} \frac{dr d\theta}{r} \le \gamma \tag{6.1}$$

where

$$\gamma = 2\pi - 4. \tag{6.2}$$

Since we will focus on u_1, for convenience we will suppress the subscript so $u(z) = u_1(z)$ and $D = D_1$.

For notational convenience we may rescale the 3 solutions $u_j(cz)/c$ $j = 1, 2, 3$ with $c > 0$ chosen so that the $f(\zeta)$ corresponding to u as in §2 satisfies (2.8).

We will consider 3 possibilities:

Case 1. $\limsup_{t\to\infty} \frac{1}{t} \int_{-t}^{t} \log |h'(is)| ds = \infty$

Case 2. $\liminf_{t\to\infty} \frac{1}{t} \int_{-t}^{t} \log |h'(is)| ds = 0$

Case 3. $0 < \liminf_{t\to\infty} \frac{1}{t} \int_{-t}^{t} \log |h'(is)| ds \le \limsup_{t\to\infty} \frac{1}{t} \int_{-t}^{t} \log |h'(is)| ds < \infty$

Henceforth we shall modify our notation, taking

$$S_\epsilon = \{-\pi/2 + \varepsilon < \arg\zeta < \pi/2 - \varepsilon\},$$

and $\varepsilon > 0$ is fixed arbitrarily small.

7. Quasiconformal mappings

We shall have occasion to view the harmonic mapping described in §2 as a quasiconformal mapping. A one to one sense preserving mapping f in a domain D is quasiconformal, if its *dilatation* $\delta = \delta(\zeta)$ defined by (cf.([1, p. 5]))

$$\delta = \left|\frac{f_{\bar\zeta}}{f_\zeta}\right| \tag{7.1}$$

satisfies

$$\sup_{\zeta\in D}\delta(\zeta) < 1. \tag{7.2}$$

The dilatation is a conformal invariant, and the inverse mapping has the same dilatation at corresponding points [1, p. 9].

We shall need a modification of the Ahlfors distortion theorem. In the classical setting we have a simply connected region G with accessible boundary points $Z_1 = X_1 + iY_1$ and $Z_2 = X_2 + iY_2$ Z_1 and Z_2. We assume that $-\infty \le X_1 = \inf \Re e\, z$ for $z \in G$ and $\infty \ge X_2 = \sup \Re e\, z$ for $z \in G$. We consider $z = x + iy$ in G with crosscuts Q_x separating Z_1 and Z_2 in G (See ([7, pp. 94-95]) for more details). Let $\Theta(x)$ be the length of Q_x. Let $w(z) = u(z) + iv(z)$ be a conformal mapping of G onto the strip $\{|v| < a/2\}$ such that Z_1 corresponds to $-\infty$ and Z_2 to $+\infty$. If $u_1(x)$ denotes the smallest value on the crosscut and $u_2(x)$ the largest, then the classical distortion theorem is as follows.

Theorem A. *If*

$$\int_{x_1}^{x_2}\frac{dx}{\Theta(x)} > 2,$$

then

$$u_1(x_2) - u_2(x_1) \ge a\int_{x_1}^{x_2}\frac{1}{\Theta(x)} - 4a.$$

For our purposes, the strip Σ_ε will be the (principal branch) logarithmic image of S_ε in the $w = \mu + i\nu$ plane and G will be the image of a fixed branch of $\log f(S_\varepsilon)$ in the $z = x + iy$ plane with f as in §2.

Let $w(z) = \log(f^{-1}(e^z))$ for the principal branch of log which then has the same dilatation as f at corresponding points. Then [13, Lemma 1],

Lemma B. *With the above notations, let R be a rectangle in the $\mu + i\nu$ plane*

$$R = \Sigma_\varepsilon \cap \ \{p \le \mu \le q\} \quad 0 < p < q,$$

and suppose that $w(z)$ has dilatation less than δ_0 in $w^{-1}(R)$. Then for x_1+iy_1 and x_2+iy_2 in $w^{-1}(R)$ $(x_1<x_2)$ and $a=\pi-\varepsilon$, if

$$\int_{x_1}^{x_2}\frac{dx}{\Theta(x)}>2\frac{1+\delta_0}{1-\delta_0}, \tag{7.3}$$

then

$$\mu_1(x_2)-\mu_2(x_1)\geq a\frac{(1-\delta_0)}{(1+\delta_0)}\int_{x_1}^{x_2}\frac{dx}{\Theta(x)}-4a. \tag{7.4}$$

8. Case 1

Let $I_n=(2^{n-1},2^n)$ for $n=1,2.....$ and $I_n=(-2^{|n|},-2^{|n|-1})$ for $n=-1,-2,....$, and $I_0=(0,1)$. Suppose that there exists a subsequence I_{n_k} and $\alpha_{n_k}\to\infty$ such that

$$\int_{I_{n_k}}\log|h'(is)|ds=\alpha_{n_k}2^{n_k}.$$

By (2.11), for $\zeta\in H$,

$$\log|h'(\zeta)|=\frac{\sigma}{\pi}\int_{-\infty}^{\infty}\frac{\log|h'(is)|ds}{|\zeta-is|^2}. \tag{8.1}$$

Using positive n_k for notational convenience, from this it follows that as $n_k\to\infty$,

$$\log|h'(\zeta)|\geq C\frac{2^{n_k}}{\sqrt{\alpha_{n_k}}}\frac{\alpha^{n_k}2^{n_k}}{2^{2n_k}}\to\infty\quad\zeta\in S_\epsilon\cap\{2^{n_k}/\sqrt{\alpha_{n_k}}\leq|\zeta|\leq 2^{n_k}\}.$$

for some constant C.

Summarizing this, we obtain sequences $\{r_n'\}$ and $\{r_n''\}$ both tending to ∞ such that $r_n''/r_n'\to\infty$, and with $T_n=S_\epsilon\cap\{r_n'\leq|\zeta|\leq r_n''\}$

$$|h'(\zeta)|\to\infty\quad\zeta\in\overline{T}_n\quad n\to\infty. \tag{8.2}$$

In order to apply (8.2), we consider f as a quasiconformal mapping in T_n. As such, by (2.2), (2.7), and (7.1) this becomes $\delta=1/|h'|^2\to 0$ in T_n by (8.2).

Let $w(z)=\log(f^{-1}(e^z))$ which has the same dilatation as f at corresponding points. Let G denote the logarithmic image of $f(S_\epsilon)$. Then w maps G into the strip $(\pi-\varepsilon)/2<\Im m\, w<(\pi+\varepsilon)/2$. Let R_n be the rectangle corresponding to T_n in the logarithmic plane with $s_{1,n}$ denoting the left side and $s_{2,n}$ the right side of R_n.

Next we take the subsets $\{G_n\}$ of G corresponding to the preimages $w^{-1}(R_n)$, and take

$$x_{1,n}=\sup_{z\in w^{-1}(s_{1,n})}\Re e\, z\ \text{ and }\ x_{2,n}=\inf_{z\in w^{-1}(s_{2,n})}\Re e\, z.$$

We note that

$$\begin{aligned}&\log|f(\zeta)|\leq x_{1,n}\quad(\zeta\in S_\varepsilon\cap\{|\zeta|=r_n'\}\\&\log|f(\zeta)|\geq x_{2,n}\quad(\zeta\in S_\varepsilon\cap\{|\zeta|=r_n''\}\end{aligned} \tag{8.3}$$

Now,

$$\int_{\log r'_n}^{\log r''_n} \frac{dx}{\Theta(x)} > \frac{1}{2\pi} \log \frac{r''_n}{r'_n}$$

which assures the condition (7.3).

Since

$$\int_{\log r'_n}^{\log r''_n} \frac{dx}{\Theta(x)} \int_{\log r'_n}^{\log r''_n} \Theta(x)dx \geq (\log \frac{r''_n}{r'_n})^2,$$

from (6.1)we have

$$\int_{\log r'_n}^{\log r''_n} \frac{dx}{\Theta(x)} > \frac{\log (r''_n(1+o(1))}{\gamma} \qquad (n \to \infty).$$

Using Lemma B we then have

$$x_{2,n} - x_{1,n} > \frac{\pi - \varepsilon}{\gamma} \log(r''_n(1+o(1))) \qquad (n \to \infty).$$

Combining (8.3) with Theorem 1 and (6.2) we then obtain a contradiction which shows that Case 1 is not possible.

9. Case 2

Having eliminated Case 1, it follows that

$$\int_{-s}^{s} \log |h'(it)|dt < Cs \tag{9.1}$$

for some constant $C > 0$.

For Case 2 we may assume that there is a sequence $\{s_n\}$ tending to ∞ such that

$$\frac{\int_{-s_n}^{s_n} \log |h'(it)|dt}{s_n} \leq \beta_n \to 0, \quad \beta_n s_n \to \infty. \tag{9.2}$$

By (9.1) an (2.11) we have

$$\log |h'(\sigma)| = \frac{\sigma}{\pi} \int_{-\infty}^{\infty} \frac{\log |h'(is)|ds}{\sigma^2 + s^2} \leq \sigma \frac{Cs}{\sigma^2 + s^2}\Big|_0^{\infty} + C\sigma \int_0^{\infty} \frac{s^2 ds}{(\sigma^2 + s^2)^2}.$$

From this it follows that

$$\log |h'(\sigma)| < C$$

and consequently

$$|h(\sigma| < e^C \sigma \tag{9.3}$$

for some constant C.

We now break the integral in (2.11) into parts and focus on the values of $\log|h'|$ on intervals of the ray $\zeta = \sigma$ with

$$\sigma \in J_n = [\beta_n^{1/2} s_n, \beta_n^{1/4} s_n].$$

Now,

$$\frac{\sigma}{\pi}\int_{-\beta^{3/4}s_n}^{\beta_n^{3/4}s_n} \frac{\log|h'(is)|ds}{\sigma^2+s^2} \le C\sigma \frac{s}{\sigma^2+s^2}\Big|_0^{\beta_n^{3/4}s_n} + C\sigma\int_0^{\beta_n^{3/4}s_n} \frac{s^2 ds}{(\sigma^2+s^2)^2}.$$

Simple computations with σ on the interval J_n show that the terms on the right tend to 0.

The same approach on $(-\infty, -s_n)\cup(s_n,\infty)$ gives

$$\frac{\sigma}{\pi}\int_{(-\infty,-s_n)\cup(s_n,\infty)} \frac{\log|h'(is)|ds}{(\sigma^2+s^2)^2} \to 0$$

for $\sigma \in J_n$ as $n \to \infty$.

Finally, using (9.2) we have

$$\frac{\sigma}{\pi}\int_{(-s_n,-\beta_n^{3/4}s_n)\cup(\beta_n^{3/4}s_n,s_n)} \frac{\log|h'(is)|ds}{\sigma^2+s^2} \le C\sigma\frac{\beta_n s_n}{\sigma^2+s^2}\Big|_{\beta_n^{3/4}s_n}^{s_n} + C\beta_n s_n \sigma\int_{\beta_n^{3/4}s_n}^{s_n} \frac{sds}{(\sigma^2+s^2)^2}$$

where terms on the right again tend to 0 for $\sigma \in J_n$.

Thus, we have

$$\log|h'(\sigma)| \to 0 \qquad \sigma\to\infty,\ \sigma\in J_n. \tag{9.4}$$

Let $\log h'(\sigma) = \varphi(\sigma) + i\psi(\sigma)$ with φ and ψ real, and by (9.4) $\varphi(\sigma)\to 0$ as $\sigma\to\infty$ in J_n. Then from (2.2) and (2.6), on the intervals J_n

$$|f(\beta_n^{1/4}s_n) - f(\beta_n^{1/2}s_n)| = |\int_{\beta_n^{1/2}s_n}^{\beta_n^{1/4}s_n} (h'(t) - 1/\bar{h}'(t))d\sigma|$$

$$= |\int_{\beta_n^{1/2}s_n}^{\beta_n^{1/4}s_n} e^{\varphi(\sigma)+i\psi(\sigma)} - e^{-\varphi(\sigma)+i\psi(\sigma)}d\sigma|$$

$$\le \int_{\beta_n^{1/2}s_n}^{\beta_n^{1/4}s_n} |e^{\varphi(\sigma)+i\psi(\sigma)} - e^{-\varphi(\sigma)+i\psi(\sigma)}|d\sigma \le \int_{\beta_n^{1/2}s_n}^{\beta_n^{1/4}s_n} \varepsilon_n d\sigma$$

where $\varepsilon_n \to 0$ as $n\to\infty$. Using this and (9.3) we deduce that

$$u(\beta_n^{1/4}s_n)/|f(\beta_n^{1/4}s_n)| > \beta_n^{1/4}/(C\beta_n^{1/2} + \varepsilon_n\beta_n^{1/4})$$

which contradicts Theorem 1. Thus, Case 2 is complete.

10. Case 3

Turning now to Theorem 2, we first have

Lemma 6. *Let f and U be as in §2, and assume the conditions of Case 3 are satisfied. Then, given $0 < \varepsilon < \pi/2$, there exists $C_2 > 0$ (depending on U, but independent of ε) such that*

$$|h'(\zeta)| \geq C_2(1 + o(1)) \exp(\varepsilon) \qquad (\zeta \to \infty,\ \zeta \in S_\varepsilon).$$

Proof. Using (2.11) and moving the "mass" to the positive y axis we may write

$$\log |h'(\zeta)| \geq \frac{\sigma}{\pi} \int_0^\infty \frac{\Psi(s)}{\sigma^2 + (s + t\sigma)^2} ds \qquad (\zeta \in S_\varepsilon), \tag{10.1}$$

where $\Psi(s) = \log |h'(is) + \log |h'(-is)|$, $|\zeta| = |\sigma + it\sigma|$ and $t = \tan(\frac{\pi}{2} - \varepsilon)$.

Integrating (10.1) by parts, the right side of (10.1) becomes

$$\frac{\sigma}{\pi} \left(\frac{\int_0^s \Psi(x)dx}{\sigma^2 + (s + t\sigma)^2} \bigg|_0^\infty + 2 \int_0^\infty \frac{(s + t\sigma) \int_0^s \Psi(x)dx}{(\sigma^2 + (s + t\sigma)^2)^2} ds \right). \tag{10.2}$$

From the elimination of Case 1, it follows that the first term in (10.2) vanishes.

Regarding the second term, using the elimination of Case 2 we can write

$$\frac{2\sigma}{\pi} \int_0^\infty \frac{(s + t\sigma) \int_0^s \Psi(x)dx}{(\sigma^2 + (s + \sigma t)^2)^2} ds \geq (1 + o(1)) \frac{2\sigma}{\pi} \int_0^\infty \frac{(s + t\sigma)Cs}{(\sigma^2 + (s + t\sigma)^2)^2} ds$$

as $\sigma \to \infty$.

Since

$$\sigma \int_0^\infty \frac{(s + t\sigma)s}{(\sigma^2 + (s + t\sigma)^2)^2} ds = \frac{1}{2} \left(\arctan(s/\sigma + t) - \frac{s\sigma}{\sigma^2 + (s + t\sigma)^2} \right) \bigg|_{s=0}^\infty = \pi/2 - \arctan(t)$$

Lemma 6 follows. □

In order to estimate Δu we use estimates from [12].

Theorem E. *There exists a constant K depending on u such that, if u is as in (1.1), and $C > 0$, then the curvature $\kappa = \kappa(C)$ of the level set $u = C$ satisfies the inequaltiy*

$$|\kappa| \leq \frac{K}{C}. \tag{10.3}$$

Proof of Theorem 2.. The curvature of a level curve through a point is given by (cf. [3, p72])

$$\kappa = \frac{u_y^2 u_{xx} + u_x^2 u_{yy} - 2u_x u_y u_{xy}}{|\nabla u|^3} \tag{10.4}$$

and by (1.1),

$$\Delta u = -u_y^2 u_{xx} - u_x^2 u_{yy} + 2u_x u_y u_{xy}. \tag{10.5}$$

Furthermore in S_ε we have by [8, (p.105)] and (2.7),

$$|1/h'(\zeta)|^2 = |a(\zeta)| = \frac{\sqrt{1 + |\nabla u(z)|^2} - 1}{\sqrt{1 + |\nabla u(z)|^2} + 1},$$

and hence,

$$|\nabla u|^2 = \left(\frac{|h'|^2 + 1}{|h'|^2 - 1}\right)^2 - 1 < \frac{4}{(|h'| - 1)^2} \tag{10.6}$$

The statement (2.9) then follows from (10.4), (10.5), (10.6) and Lemma 6.

Regarding (2.10), starting with (1.2), we observe that for large $z \in f(S_{\pi/4})$ and $v(z)$ as in (2.12), then $v(z) < u(z) < \kappa_2|z|$. For $\alpha > \pi/4$, $v(z) > u(z)$ in $f(S_{\pi/4})$. Since $|\nabla v(z)| < 1$ we have $|v(z)| < \kappa_2|z| + 2\pi|z|$. Thus, when $\tan\alpha = (\kappa_2 + 2\pi)/\varepsilon$, (2.10) holds.

□

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